Effect of a Tangential Electric Field on the Second Harmonic Resonance in Kelvin-Helmholtz Flow

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Z. Naturforsch. **51a**, 10–16 (1996); received November 24, 1995

The stability of Kelvin-Helmholtz flow at the second harmonic resonance between two marginally unstable modes is investigated in the presence of a tangential electric field. We obtain coupled amplitude equations which are examined for the stability characteristics. Numerical integration of the equations reveals that at a certain determinate time, the amplitudes of both the fundamental mode and the second harmonic mode start increasing rapidly, resulting in what is termed as "Explosive Instability". It is demonstrated that for a fluid of given density and dielectric constant, the electric field plays an important role, and the explosive instability sets in much earlier as the electric field increases.

1. Introduction

The effect of an electric field on the motion of fluids has been studied by a number of researchers since the pioneering work of Lord Rayleigh [1] and Stokes [2]. The electrohydrodynamic (EHD) body force is nonlinear, and therefore the phenomenon cannot be discussed in its entirety by a linear theory. Most of the work on nonlinear EHD concerns conducting fluids (see: Melcher [3], Melcher and Taylor [4], Michael [5], Kant et al. [6]), where the dielectric constant plays no role.

Gross and Porter [7] observed in their experiments that instability occurs when a thermally and gravitationally stable stratification of a fluid is subjected to an electric field. They explained this instability by the suggestion that, when a nonuniform field is applied to an inhomogeneous fluid with a nonuniform dielectric constant, regions of lower dielectric constant will experience a force directed towards regions of lower field strength. This leads to instability if the dielectric constant varies because of heating or any source of inhomogeneity. Later it was found that the temperature dependence of the dielectric constant has no significant effect on the fluid layer, and that the stabilizing or destabilizing effects of the electric field depend on whether the conductivity is a linear function of temperature or a quadratic function.

Reprint requests to Prof. S.K. Trehan.

Mohamed and Shehawey [8] observed that the stability criterion for a surface separating two uncharged homogeneous dielectrics should be affected by the jump in the dielectric constant and electric field across the surface. Such effects can only be understood by nonlinear analysis, as linear analysis fails to predict them. This led to the study on the nonlinear EHD stability of a surface separating two dielectric fluids in the presence of an electric field.

Cairns [9] has shown that insight into the linear and nonlinear instability of certain parallel flows can be got by the concept of negative wave energy. For such flows Cairns [9] found that nonlinear three-wave resonant instabilities exist, in which the total wave energy is conserved but the three wave amplitudes can grow simultaneously. Such a phenomenon is known a explosive instability because the theoretical solutions attain infinite amplitudes after a finite time. Of course, the weakly nonlinear theory must break down before this singularity is attained. This situation is a well known phenomenon in plasma physics, but despite its likely importance, no example had been recognised in fluid mechanics until the work of Cairns [9].

Craik and Adams [10] analysed the linear stability and nonlinear three wave resonance in a three layer fluid flow with stepwise velocity and density profiles. They confirmed the existence of an explosive instability in which all three waves of the resonant triad grow simultaneously while the total wave energy is conserved. The interaction equations, truncated at second

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order, have solutions which develop a singularity after a finite time. They pointed out that there also exists a special case when a wave and its own second harmonic form a resonant triad of the form $K_1 = 2K_2$ and $\omega_1 = 2\omega_2$. Such cases have been investigated extensively by McGoldrick [11] and Nayfeh [12] for waves on a single interface.

Nayfeh et al. [13] investigated the nonlinear waves on the interface of two incompressible inviscid fluids of different densities and arbitrary surface tension, using the method of multiple scales. They obtained a second order expansion for wave numbers near the second harmonic resonant wave number, for which the fundamental and its second harmonic have the same phase velocity and found that this resonance does not lead to instabilities.

Murakami [14] investigated the second harmonic resonance between two marginally unstable modes for the Kelvin-Helmholtz flow and considered the neighbourhood of the marginally neutral curve by letting the bifurcation parameter, U, be $U_m \pm \varepsilon^2$, $\varepsilon \ll 1$, where U_m corresponds to a point of marginal stability. In order to separate the rapid change of phase and the slow change of amplitude of the carrier wave, he introduced the multiple scales in space and time variables. The surface elevation and all physical parameters are expanded as an asymptotic expansion with lowest order $o(\varepsilon^2)$ (see: Weissman [15]). This leads to the conclusion that amplitudes are of the order $o(\varepsilon^2)$, which is different from the second harmonic resonance between two neutral modes where amplitudes are of the order $o(\varepsilon)$. Murakami [14] obtained coupled amplitude equations which depend on both time and space. Bounded solutions of these equations without spatial dependence reveal that explosive instability occurs in the system. Thus the second harmonic resonance induces this instability. This is quite different from the three layer model introduced by Craik and Adams [10], in which the explosive instability can take place owing to the interaction between neutral modes.

The effect of a tangential field on the instability of EHD flows has drawn considerable attention in recent years (see: Mohamed and Shehawey [16], Elhefnawy [17]). For two semi-infinite dielectric fluids separated by the plane $y = 0$; it was found that the analysis for the normalized wave number $k = [(1 - \varrho)/2]^{1/2}$, where ϱ denotes the ratio of the densities of the two fluids involved, breaks down. In this case the coefficients of the amplitude equation have a singularity, and hence it diverges. The appearance of such a singularity manifests as the phenomena of second harmonic resonance.

In this work we consider the weakly nonlinear theory of resonant wave interactions at a charge free surface separating two semi-infinite dielectric streaming fluids influenced by a tangential electric field. In section 3 we obtain the conditions under which nonlinear instability can occur. The numerical solutions of the coupled dynamical equations governing the amplitudes of the interacting waves reveals that both modes diverge at finite time which depends on the given initial conditions and parameters, thus leading to an explosive instability.

2. Formulation of the Problem

We consider tangential electric field instability of two semi-infinite dielectric fluids separated by the plane $y = 0$. The fluids are assumed to be incompressible, inviscid with densities $\varrho^{(1)}$, $\varrho^{(2)}$ and streaming with velocities $U^{(1)}$ and $U^{(2)}$ along the positive x-direction. The superscripts 1 and 2 refer to the region $y < 0$ and $y > 0$, respectively; $e^{(1)}$ and $e^{(2)}$ are the dielectric constants of the fluids. The gravitational force per unit mass is $g(0, -1, 0)$, whereas the electric fields $E_0^{(1)}$ and $E_0^{(2)}$ act tangential to the interface. The interface is taken to be $S(x, t) = y - \eta(x, t)$, where $\eta(x, t)$ is the surface elevation of the free surface from the unperturbed level. We shall assume that there are no charges at the surface of separation in the equilibrium state, and therefore the electric displacement is continuous at the interface. The flow field is assumed to be irrotational. The velocity potential $\Phi^{(j)}(x, y, t)$ and the electrostatic potential $\Psi^{(j)}(x, y, t)$ ($j = 1, 2$ in the appropriate region) satisfy Laplace's equation. At a large distance from the interface, the normal component of velocity and electric field vanishes. At the free surface $y = \eta(x, t)$ the boundary conditions are:

(i) the interface is moving with the fluid,
(ii) the tangential component of the electric field is continuous,
(iii) the normal electric displacement is continuous, since there are no surface charges at the interface,
(iv) the normal hydrodynamical stress is balanced by the normal electric stress.

These boundary conditions are expressed at $y = \eta(x, t)$ as

$$\Phi_y^{(j)} = \eta_t + U^{(j)} \eta_x + \eta_x \Phi_x^{(j)}, \quad j = 1, 2 \tag{1}$$

$$\eta_x [\![\psi_y]\!] + [\![\psi_x]\!] = 0, \tag{2}$$

$$\eta_x [\![e(E_0 - \psi_x)]\!] + [\![e\psi_y]\!] = 0, \tag{3}$$

$$\begin{aligned}&[\![\varrho\{\Phi_t + g\eta + U\Phi_x + \tfrac{1}{2}(\Phi_x^2 + \Phi_y^2)\}]\!]\\ &\quad + [\![e\{\psi_y(\psi_y - 2\eta_x\psi_x) - \tfrac{1}{2}(\psi_x^2 + \psi_y^2)\}]\!]\\ &\quad + [\![e E_0\{\psi_x(1 - 2\eta_x^2) + 2\eta_x\psi_y + e E_0^2 \eta_x^2\}]\!]\\ &\qquad = T\eta_x^2(1 + \eta_x^2)^{3/2},\end{aligned} \tag{4}$$

where $[\![\phi]\!] = \phi^{(1)} - \phi^{(2)}$ represents the jump in ϕ across the interface $y = 0$ and T is the surface tension. We assume that the lower fluid is at rest ($U^{(1)} = 0$) and the upper fluid is moving with a uniform velocity U ($U^{(2)} = U$).

To describe the resonance interaction between two marginally unstable wave modes of small but finite amplitude, we use the method of multiple scales as formulated by Weissman [12] and Murakami [11]. We introduce a small parameter $\varepsilon = (U/U_m - 1)^{1/2}$, where U_m is the marginally stable velocity of the fluid. This parameter represents the departure of the system from the bifurcation value U_m. This bifurcation value is such that the difference between the original velocity U and the final velocity U_m is of order ε^2; $\varepsilon \ll 1$. Thus, we discuss here the stability of the system in the neighbourhood of the neutral curve by setting the bifurcation parameter U as $U_m(1 \pm \varepsilon^2)$. For a significant interaction at such a resonance, and also in order to represent the rapid change of phase and the slow change of amplitude of the carrier wave, we introduce multiple scales in space and time as $x_n = \varepsilon^n x$, $t_n = \varepsilon^n t$. Since we wish to describe the nonlinear interaction in a slightly unstable region, we expand the various physical quantities as

$$\xi(x_0, x_1, x_2, y, t_1, t_2; \varepsilon) = \varepsilon^2 \xi_1 + \varepsilon^3 \xi_2 + \varepsilon^4 \xi_3 + \ldots, \tag{5}$$

$$U = U_m \pm \varepsilon^2 U_m. \tag{6}$$

The differential operators in the various equations are expanded as follows:

$$\frac{\partial}{\partial\alpha} = \frac{\partial}{\partial\alpha_0} + \varepsilon\frac{\partial}{\partial\alpha_1} + \varepsilon^2\frac{\partial}{\partial\alpha_2} + \ldots, \tag{7}$$

where α is any of the variables x or t. The approximate solution of the system of equations can be obtained by using (5)–(7). On equating the coefficients of like powers in ε, we obtain linear as well as successive higher order equations (see: Appendix). It is sufficient to consider only the solution up to $O(\varepsilon^4)$ as far as the lowest significant approximation is concerned.

3. Linear and Nonlinear Analysis

We find that the $O(\varepsilon^2)$ problem is equivalent to the linear problem. If we neglect the second harmonic resonance, then the linear theory yields the dispersion relation

$$\begin{aligned}D(\omega, k) &= \varrho^{(1)}\omega^2 + \varrho^{(2)}(\omega - kU)^2\\ &\quad - gk(\varrho^{(1)} - \varrho^{(2)}) - Tk^3 - \frac{e^{(2)}k^2E_0^2(1-e)^2}{1+e} = 0.\end{aligned} \tag{8}$$

Here ω, k denote the frequency and the wave number of the fundamental mode only. $e = e^{(1)}/e^{(2)}$, and at the interface $E_0^{(1)} = E_0^{(2)} = E_0$. It is clear from the dispersion relation that it does not depend on the sign of $(1 - e)$. Equation (8) yields the critical wave number

$$k_c = \left[\frac{g}{T}(\varrho^{(2)} - \varrho^{(1)})\right]^{1/2} [\sinh(\theta_E) + \cosh(\theta_E)], \tag{9}$$

where

$$\sinh(\theta_E) = \frac{\alpha_E}{2}[Tg(\varrho^{(2)} - \varrho^{(1)})]^{-1/2}, \tag{10}$$

$$\alpha_E = \frac{\varrho^{(2)}U^2}{(1+\varrho)} - \frac{e^{(2)}E_0^2(1-e)^2}{(1+e)}, \quad \varrho = \frac{\varrho^{(2)}}{\varrho^{(1)}}. \tag{11}$$

Thus, the interface is linearly stable when $k > k_c$. It is found that the electric field is strictly destabilizing in the linear analysis. To describe the second harmonic resonance on the marginally neutral curve, it is required that in addition to the fundamental mode $A_1(x_1, x_2, t_1, t_2)$ we must take into account the harmonic mode $A_2(x_1, x_2, t_1, t_2)$, where $\omega_2 = 2\omega_1 + O(\varepsilon)$ and $k_2 = 2k_1 + O(\varepsilon)$. The second-order problem, governed by (A.2)–(A.5), admits the solution

$$\begin{aligned}\eta_1 &= \sum_{n=1}^{2} A_n e^{i\theta_n} + \text{c.c.},\\ \Phi_1^{(1)} &= -i\sum_{n=1}^{2}\left(\frac{\omega_n}{k_n}\right) A_n e^{(i\theta_n + k_n y)} + \text{c.c.},\\ \Phi_1^{(2)} &= i\sum_{n=1}^{2}(\omega_n - k_n U_m) A_n e^{(i\theta_n - k_n y)} + \text{c.c.},\\ \psi_1^{(j)} &= \frac{i e^{(2)}(1-e)}{1+e} E_0^{(j)} \sum_{n=1}^{2} A_n e^{\psi_{nj}} + \text{c.c.}, \quad j = 1, 2,\end{aligned} \tag{12}$$

where $\theta_n = k_n x_0 - \omega_n t_0$, $\psi_{nj} = i\theta_n - (-1)^j k_n y$. The uniformly valid solution, when substituted in (A.5), admits the dispersion relation $D(\omega_n, k_n) = 0$. The resonant harmonic wave number can be obtained from the dispersion relation by letting $\omega_2 = 2\omega_1$ and $k_2 = 2k_1$. We get the marginally stable wave number $k_1 = [g(\varrho^{(1)} - \varrho^{(2)})/(2T)]^{1/2} = k_m$. The value of the fluid velocity U at the marginally neutral curve is

$$U_m = \left[\frac{k_m T(1+\varrho)}{\varrho^{(2)}} + \frac{g(1-\varrho^2)}{k_m \varrho} + \frac{e^{(2)} E_0^2 (1-e)^2 (1+\varrho)}{(1+e)\varrho^{(2)}}\right]^{1/2}. \tag{13}$$

The system is unstable for $U \geq U_m$. To examine the nonlinear coupling due to second harmonic resonance we now consider the third-order problem. On substituting the solutions of the second-order problem into (A.6)–(A.10) we obtain the uniformly valid solutions of the problem. These solutions lead to the following solvability condition on the marginally neutral curve:

$$\frac{\partial A_n}{\partial x_1} = 0, \quad n = 1, 2. \tag{14}$$

This condition reveals that the amplitudes A_n are independent of the faster scale x_1, but may depend upon x_2, t_1 and t_2. We now substitute the second and third order solutions into the fourth-order problem governed by (A.11)–(A.16). If we consider the terms upto $\exp(i\theta_n)$ and ignore the higher order terms, which are of no interest here, we obtain

$$\Delta\Phi_3^{(j)} = (-1)^j \sum_{n=1}^{2}\left[2\omega_n \nabla_{2,x} A_n + i\frac{\omega_n}{k_n}\{1 - 2(-1)^{j+1} y k_n\}\nabla_{1,x}^2 A_n + 2\nabla_{1,t}\nabla_{1,x} A_n\right] e^{i\psi_{nj}}$$
$$- \lambda_j U_m \sum_{n=1}^{2} [2k_n \nabla_{2,x} A_n + i(2yk_n - 1)\nabla_{1,x}^2 A_n] e^{i\psi_{nj}} + \ldots, \quad j = 1, 2, \tag{15}$$

$$\Delta\psi_3^{(j)} = \frac{E_0(1-e)}{1+e} \sum_{n=1}^{2} [2k_n \nabla_{2,x} A_n - i\{1 + 2(-1)^{j+1} y k_n\}\nabla_{1,x}^2 A_n] e^{i\psi_{nj}} + \ldots, \quad j = 1, 2, \tag{16}$$

$$D(\Phi_3^{(1)}, \eta_3) = \sum_{n=1}^{2} [\nabla_{2,t} A_n] e^{i\theta_n} + i\omega_1 k_1 \{3A_2 \bar{A}_1 e^{i\theta_1} + 2A_1^2 e^{i\theta_2}\} + \ldots, \tag{17}$$

$$D(\Phi_3^{(2)}, \eta_2) - U_m \nabla_{0,x}\eta_3 = \sum_{n=1}^{2} [\nabla_{1,t} A_n + U_m \nabla_{2,x} A_n \pm i k_n U_m A_n] e^{i\theta_n}$$
$$- i k_1(\omega_1 - k_1 U_m)\{3A_2 \bar{A}_1 e^{i\theta_1} + 2A_1^2 e^{i\theta_2}\} + \ldots, \tag{18}$$

$$L_0(\psi_3^{(1)}, \psi_3^{(2)}) = \frac{2k_1^2 E_0(1-e)}{(1+e)} [A_2 \bar{A}_1 e^{i\theta_1} + 2A_1^2 e^{i\theta_2}] + \ldots, \tag{19}$$

$$R(\psi_2^{(1)}, \psi_3^{(2)}, y) - e^{(2)}(1-e)E_0 \nabla_{0,x}\eta_3 = e^{(2)}(1-e)E_0 \sum_{n=1}^{2} [\nabla_{2,x} A_n] e^{i\theta_n}$$
$$+ \frac{ik_1^2 e^{(2)}(1-e)^2 E_0}{(1+e)} [3A_2 \bar{A}_1 e^{i\theta_1} + 2A_1^2 e^{i\theta_2}\} + \ldots, \tag{20}$$

$$M_0(\Phi_3^{(1)}, \Phi_3^{(2)}) - \varrho^{(2)} U_m \nabla_{0,x}\Phi_3^{(2)} + E_0 R(\psi_3^{(1)}, \psi_3^{(2)}, x_0) + Q\eta_3$$
$$= \varrho^{(1)} \sum_{n=1}^{2}\left[-\frac{(1+\varrho)}{k_n}\nabla_{1,t}^2 A_n - \frac{1}{k_n^2}\{(1+\varrho)\omega_n + \varrho U_m k_n\}\nabla_{1,x}\nabla_{1,t} A_n + \left\{\frac{T}{\varrho^{(1)}} - \frac{\varrho U_m \omega_n}{k_n^2}\right\}\nabla_{1,x}^2 A_n\right.$$
$$+ \frac{i}{k_n}\{(1+\varrho)\omega_n - \varrho U_m k_n\}\nabla_{2,t} A_n + i\left\{\frac{\varrho U_m}{k_n}(\omega_n - k_n U_m) + \frac{E_0^2 e^{(2)}(1-e)^2}{\varrho^{(1)}(1+e)} + \frac{2Tk_n}{\varrho^{(1)}}\right\}\nabla_{2,x} A_n$$
$$\left.\mp \varrho U_m(\omega_n - k_n U_m) A_n\right] e^{i\theta_n} + \{\varrho^{(1)}\omega_1^2 - \varrho^{(2)}(\omega_1 - k_1 U_m)^2\}\{A_2 \bar{A}_1 e^{i\theta_1} + A_1^2 e^{i\theta_2}\}$$
$$+ e^{(2)} E_0^2 k_1^2 (1-e)\{A_2 \bar{A}_1 e^{i\theta_1} + 2e(2-e) A_1^2 e^{i\theta_2}\}. \tag{21}$$

The various symbols used in (15)–(21) are defined in the Appendix. After some straightforward algebra, we find the following conditions which are necessary for having a uniformly valid solution:

$$P_n \frac{\partial^2 A_n}{\partial t_1^2} + i Q_n \frac{\partial A_n}{\partial x_2} = \pm R_n A_n + N_n A_2 \bar{A}_1 + M_n A_1^2, \quad n = 1, 2, \tag{22}$$

where

$$\begin{aligned} P_n &= -\frac{\varrho^{(1)} + \varrho^{(2)}}{k_n}, \\ Q_n &= \left[\left(\frac{\omega_n}{k_n}\right)^2 (\varrho^{(1)} + \varrho^{(2)}) - \varrho^{(2)} U_{\mathrm{m}}^2 + 2 T k_n + \frac{e^{(2)} E_0^2 (1-e)^2}{(1+e)}\right], \\ R_n &= 2 \varrho^{(2)} k_n (\omega_n - k_n U_{\mathrm{m}}), \\ N_1 &= 4 \varrho^{(1)} \omega_1^2 - 4 k_1 g (\varrho^{(1)} - \varrho^{(2)}) + 2 T k_1^3 + \frac{4 k_1^2 e^{(2)} E_0^2 (1-e)^2}{1+e}, \\ N_2 &= 0, \quad M_1 = 0, \quad M_2 = \frac{N_1}{2}. \end{aligned} \tag{23}$$

Here the $\pm$ signs of the linear terms coincide with $U = U_{\mathrm{m}} \pm \varepsilon^2$. It appears that it is difficult to solve the coupled dynamical equations (22). We find that these equations reduce to ordinary differential equations if we neglect the space dependence due to variable x_2. We apply the fourth-order Runge-Kutta method to integrate numerically the dynamical equations assuming the spatial dependence to be negligible. We consider fluids with different densities and dielectrics by taking different values of E_0. We have chosen the

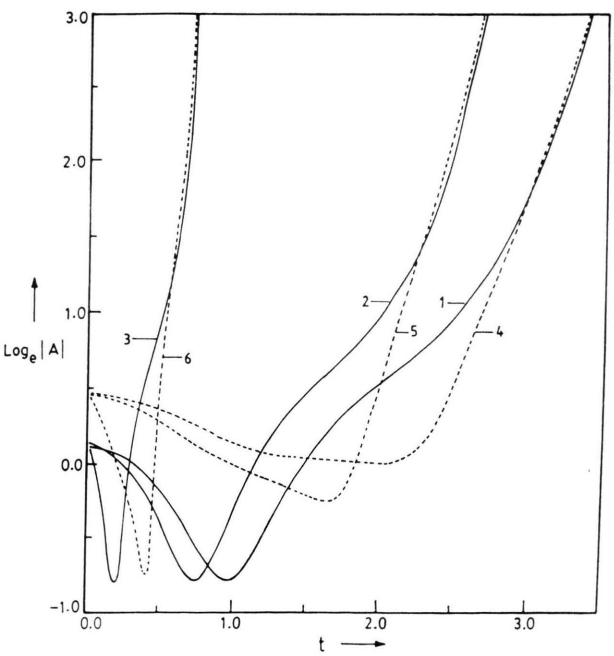

Fig. 1. The behaviour of amplitudes $A(t)$ as a function of time t for $\varrho^{(1)} = 1.02500$, $\varrho^{(2)} = 0.00126$, $e^{(1)} = 79.63000$ and $e^{(2)} = 1.00059$. The curves 1, 2, and 3 represent graphs of $\mathrm{Log}_e |A_1|$, while the curves 4, 5, and 6 represent the graphs of $\mathrm{Log}_e |A_2|$ for $E_0 = 0.0$, 0.09795 and 0.59795, respectively.

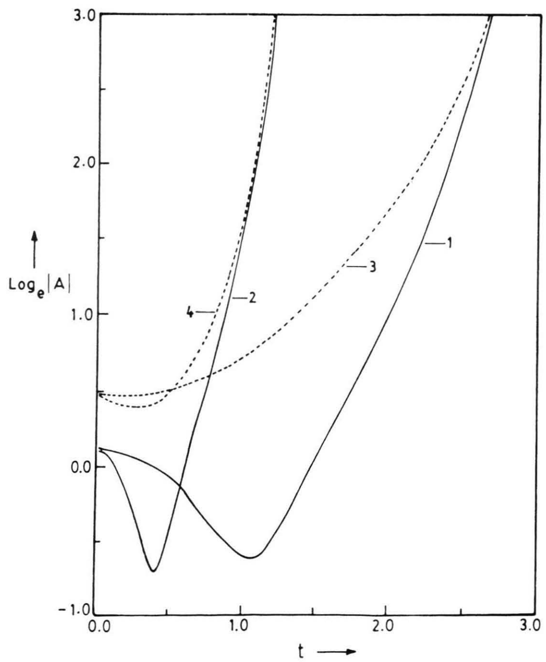

Fig. 2. The behaviour of amplitudes $A(t)$ as a function of time t for a given electric field parameter ($E_0 = 2.59795$). The curves 1 and 2 represent graphs of $\mathrm{Log}_e |A_1|$, while the curves 3 and 4 represent the graphs of $\mathrm{Log}_e |A_2|$ for $\varrho = 0.960$, $e^{(2)}/e^{(1)} = 17.0514$ and $\varrho = 0.878$, $e^{(2)}/e^{(1)} = 37.3850$, respectively.

initial conditions (see Murakami [14])

$$\begin{aligned} &\mathrm{Re}(A_1) = 1; && \mathrm{Im}(A_1) = 0.5; \\ &\mathrm{Re}(A_2) = -0.5; && \mathrm{Im}(A_2) = -1.5. \end{aligned} \tag{24}$$

The initial speeds of A_1 and A_2 are taken to be zero. We found similar behaviour for different sets of initial data. Typical solutions are shown in Figs. 1 and 2. In all cases, the results reveal that the solutions diverge at a finite time. Thus the amplitudes of both the fundamental mode and the second harmonic mode start increasing sharply after a finite time resulting in what is termed as Explosive Instability.

4. Discussion

We consider the flow of air over sea water. The various constants are taken as $\varrho^{(1)} = 1.02500$, $\varrho^{(2)} = 0.00126$, $e^{(1)} = 70.6300$ and $e^{(2)} = 0.00059$. In Fig. 1, we have plotted the time development of the amplitudes for different values of E_0, while in Fig. 2 we have plotted similar diagrams for different density and dielectric constant ratios and $E_0 = 2.5975$. These diagrams reveal that the amplitudes grow unboundedly in finite time, thus leading to explosive instability near the second harmonic resonance. It is found that for a fluid of given density and dielectric constant, the electric field plays an important role in the stability characteristics. The explosive instability sets in much earlier when the electric field increases. On the other hand, the explosive instability takes place at a slower rate for a fluid of higher density (or lower dielectric constant) ratio. These results are similar to those obtained by Gill and Trehan [18] in their studies for Kelvin-Helmholtz flow in hydromagnetics.

Acknowledgement

We are grateful to Gurpreet K. Gill for some useful discussion. SKT acknowledges the financial assistance from Council of Scientific and Industrial Research.

Appendix

We define the various linear operators as

$$\Delta = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y^2}, \quad D(r,s) = \frac{\partial r}{\partial y} - \frac{\partial s}{\partial t_0}, \quad \nabla = \frac{\partial}{\partial y}, \quad \nabla_{j,r} = \frac{\partial}{\partial r_j}, \quad L_j(r,s) = \frac{\partial r}{\partial x_j} - \frac{\partial s}{\partial x_j},$$

$$R(r,s,p) = e^{(1)}\frac{\partial r}{\partial p} - e^{(2)}\frac{\partial s}{\partial p}, \quad M_j(r,s) = \varrho^{(1)}\frac{\partial r}{\partial t_j} - \varrho^{(2)}\frac{\partial s}{\partial t_j},$$

$$U(r,s,p) = \frac{\partial r}{\partial p_0} + \frac{\partial s}{\partial p_1}, \quad V(r,s,p) = \frac{\partial r}{\partial p_1} + \frac{\partial s}{\partial p_2},$$

$$S(r,s,t,p) = \frac{1}{2}\left[t^{(2)}\left(\frac{\partial s}{\partial p}\right)^2 - t^{(1)}\left(\frac{\partial r}{\partial p}\right)^2\right], \quad Q = (\varrho^{(1)} - \varrho^{(2)})g - T\frac{\partial^2}{\partial x_0^2}. \tag{A.1}$$

The second-order problem $o(\varepsilon^2)$:

$$\Delta\kappa_1^{(j)} = 0; \quad j = 1, 2, \quad \kappa \text{ is } \Phi \text{ and } \psi \tag{A.2}$$

with the boundary conditions at $y = 0$

$$D(\Phi_1^{(1)}, \eta_1) = 0, \quad D(\Phi_1^{(2)}, \eta_1) - U_m \nabla_{0,x}\eta_1 = 0, \tag{A.3}$$

$$L_0(\psi_1^{(1)}, \psi_1^{(2)}) = 0, \quad R(\psi_1^{(1)}, \psi_1^{(2)}, y) + (e^{(1)} - e^{(2)})E_0 \nabla_{0,x}\eta_1 = 0, \tag{A.4}$$

$$M_0(\Phi_1^{(1)}, \Phi_1^{(2)}) - \varrho^{(2)} U_m \nabla_{0,x}\Phi_1^{(2)} + E_0 R(\psi_1^{(1)}, \psi_1^{(2)}, x) + Q\eta_1 = 0. \tag{A.5}$$

The third-order problem $o(\varepsilon^3)$:

$$\Delta\kappa_2^{(j)} = -2\nabla_{0,x}\nabla_{1,x}\kappa_1^{(j)}; \quad j = 1, 2, \quad \kappa \text{ is } \Phi \text{ and } \Psi \tag{A.6}$$

with the boundary conditions at $y=0$

$$D(\Phi_2^{(1)}, \eta_2) = \nabla_{1,t}\eta_1\,, \quad D(\Phi_2^{(2)}, \eta_2) = \nabla_{1,t}\eta_1 + U_{\mathrm{m}} U(\eta_2, \eta_1, x)\,, \tag{A.7}$$

$$L_0(\psi_2^{(1)}, \psi_2^{(2)}) = L_1(\psi_1^{(2)}, \psi_1^{(1)})\,, \tag{A.8}$$

$$R(\psi_2^{(1)}, \psi_2^{(2)}, y) + (e^{(1)} - e^{(2)}) E_0 U(\eta_2, \eta_1, x) = 0\,, \tag{A.9}$$

$$\begin{aligned} M_0(\Phi_2^{(1)}, \Phi_2^{(2)}) + E_0 R(\psi_2^{(1)}, \psi_2^{(2)}, x_0) + Q\eta_2 = &- M_1(\Phi_1^{(1)}, \Phi_1^{(2)}) - E_0 R(\psi_1^{(1)}, \psi_1^{(2)}, x_1) \\ &+ \varrho^{(2)} U_{\mathrm{m}} U(\Phi_2^{(2)}, \Phi_1^{(2)}, x) + 2T\nabla_{0,x}\nabla_{1,x}\eta_1\,. \end{aligned} \tag{A.10}$$

The fourth-order problem $o(\varepsilon^4)$:

$$\Delta\kappa_3^{(j)} = -2\nabla_{0,x} V(\kappa_2^{(j)}, \kappa_1^{(j)}, x) - \nabla_{1,x}^2 \kappa_1^{(j)}; \quad j = 1, 2, \quad \kappa \text{ is } \Phi \text{ and } \Psi \tag{A.11}$$

with the boundary conditions at $y=0$

$$D(\Phi_3^{(1)}, \eta_3) = V(\eta_2, \eta_1, t) - \eta_1 \nabla^2 \Phi_1^{(1)}\,, \tag{A.12}$$

$$D(\Phi_3^{(2)}, \eta_3) - U_{\mathrm{m}}\nabla_{0,x}\eta_3 = V(\eta_2, \eta_1, t) + U_{\mathrm{m}}\{V(\eta_2, \eta_1, x) \pm \nabla_{0,x}\eta_1\} + (\nabla_{0,x}\eta_1)(\nabla_{0,x}\Phi_1^{(2)}) - \eta_1\nabla^2\Phi_1^{(2)}\,, \tag{A.13}$$

$$L_0(\psi_3^{(1)}, \psi_3^{(2)}) = L_1(\psi_2^{(2)}, \psi_2^{(1)}) + L_2(\psi_1^{(2)}, \psi_1^{(1)}) + \eta_1 \nabla L_0(\psi_1^{(2)}, \psi_1^{(1)}) + (\nabla_{0,x}\eta_1)(\nabla\psi_1^{(2)} - \nabla\psi_1^{(1)})\,, \tag{A.14}$$

$$\begin{aligned} R(\psi_3^{(1)}, \psi_3^{(2)}, y) + (e^{(1)} - e^{(2)}) E_0 (\nabla_{0,x}\eta_3 + V(\eta_2, \eta_1, x)) = &\ \eta_1\{e^{(2)}\nabla^2\psi_2^{(1)} - e^{(1)}\nabla^2\psi_1^{(1)}\} \\ &+ (\nabla_{0,x}\eta_1)(e^{(1)}\nabla_{0,x}\psi_1^{(1)} - e^{(2)}\nabla_{0,x}\psi_2^{(1)})\,, \end{aligned} \tag{A.15}$$

$$\begin{aligned} M_0(\Phi_3^{(1)}, \Phi_3^{(2)}) - \varrho^{(2)} U_{\mathrm{m}}\nabla_{0,x}\Phi_3^{(2)} + E_0 R(\psi_3^{(1)}, \psi_3^{(2)}, x_0) + Q\eta_3 = &- M_1(\Phi_2^{(1)}, \Phi_2^{(2)}) - M_2(\Phi_1^{(1)}, \Phi_1^{(2)}) \\ &- \eta_1 \nabla M_0(\Phi_1^{(1)}, \Phi_1^{(2)}) + \varrho^{(2)} U_{\mathrm{m}}(V(\Phi_2^{(2)}, \Phi_1^{(2)}, x) + \eta_1\nabla\nabla_{0,x}\Phi_1^{(2)} \pm \nabla_{0,x}\Phi_1^{(2)}) \\ &- E_0 R(\psi_2^{(1)}, \psi_2^{(2)}, x_1) - E_0 R(\psi_1^{(1)}, \psi_1^{(2)}, x_2) - E_0\eta_1\nabla R(\psi_1^{(1)}, \psi_1^{(2)}, x_0) \\ &- 2E_0(\nabla_{0,x}\eta_1) R(\psi_1^{(1)}, \psi_1^{(2)}, y) + (e^{(2)} - e^{(1)})\{E_0\nabla_{0,x}\eta_1\}^2 + S(\Phi_1^{(1)}, \Phi_1^{(2)}, \varrho, x_0) \\ &+ S(\Phi_1^{(1)}, \phi_1^{(2)}, \varrho, y) + S(\psi_1^{(1)}, \psi_1^{(2)}, e, x_0) + S(\psi_1^{(1)}, \psi_1^{(2)}, e, y) \\ &+ T\{\nabla_{1,x}^2\eta_1 + 2\nabla_{0,x}V(\eta_2, \eta_1, x)\}\,. \end{aligned} \tag{A.16}$$

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